

SUSY Ward identities for multi-gluon helicity amplitudes with massive quarks

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Abstract

We use supersymmetric Ward identities to relate multi-gluon helicity amplitudes involving a pair of massive quarks to amplitudes with massive scalars. This allows to use the recent results for scalar amplitudes with an arbitrary number of gluons obtained by on-shell recursion relations to obtain scattering amplitudes involving top quarks.

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1 Introduction

Recently, a number of efficient methods for the calculation of helicity amplitudes in QCD have been introduced, motivated by the relationship of QCD amplitudes to twistor string theory [1]. In the Cachazo - Svrček - Witten (CSW) construction [2], tree level QCD amplitudes are constructed from vertices that are off-shell continuations of maximal helicity violating (MHV) amplitudes [3], connected by scalar propagators. This formalism has been extended to massless quarks [4] and loop diagrams [5]. Subsequently a set of recursion relations has been found [6] that involve only on-shell amplitudes with shifted, complex external momenta. These methods extend earlier recursive formulations [7–9] which are based on currents with one off-shell leg. Recently a derivation of the CSW rules from the on-shell recursion relations has been given [10] and proofs of the on-shell recursion relations using conventional Feynman diagrams have appeared [11, 12]. Another elegant technique applicable to scattering amplitudes with massless particles is the use of supersymmetric Ward identities (SWIs) [13–15] to relate helicity amplitudes involving massless quarks to those involving massless scalars or to purely gluonic ones [14, 16–18].

In the light of the major role played by top quark physics at the LHC, it would be desirable to extend such methods also to multi-parton processes involving massive quarks. For instance, top-pair production with one or two additional jets contributes to the background for Higgs production by vector boson fusion or in association with a top quark pair [19]. Tree amplitudes with a top pair and additional jets are also required for the next-to leading order corrections to top pair plus jet production [20]. Helicity amplitudes for a top pair plus three gluons can be found in [21]. Tree level helicity amplitudes with massive quarks are also needed within the unitarity method [22] for one-loop amplitudes, where a heavy quark is circulating in the loop. In this context, four point amplitudes have been computed in [23] and four and five point amplitudes have been obtained from on-shell recursion relations in [24].

The on-shell recursion relations are also applicable to massive particles [25, 26] and closed expressions for some sets of amplitudes with massive scalars and an arbitrary number of gluons have recently been found [27]. In the MHV approach it has been possible to include single external massive gauge bosons or Higgs bosons [28]. For massive quarks, the MHV approach cannot be used in a straightforward way since there are nonvanishing helicity amplitudes that go to zero in the massless limit where less than two partons have a helicity opposite to the remaining ones. In the on-shell approach only four point functions with massive quarks have been computed so far [26].

As a first step to extend the formalism of [2] towards massive quarks, we have reformulated in [29, 30] the Feynman rules of QCD including massive quarks so that only scalar propagators and a set of cubic and quartic primitive vertices without Lorentz or spinor indices appear. The cubic primitive vertices in this approach can also be used directly in the on-shell recursion relations.

Recently, a particular set of multi-gluon helicity amplitudes with a pair of massive quarks has been calculated in [31]. These results should be related through supersymmetric relations to the one obtained in [27] for multi-gluon helicity amplitudes with a

pair of massive scalars. In this paper we work out the explicit transformation laws. The derivation follows the lines of the original paper [13]. There are several motivations for reconsidering the supersymmetric transformation laws: First of all, by taking into account recent developments in the definition of off-shell polarization spinors, we obtain short and elegant results. Secondly, the results allow us to relate amplitudes with massive scalars to amplitudes with massive quarks. Since there are more results available for all-multiplicity amplitudes with massive scalars, this is useful for top quark physics. Finally, we are very careful to keep internally all signs correct. This is an issue since there are different sign conventions in the supersymmetric and QCD community. The final result can of course easily be related to any other sign convention. Furthermore we have to respect the relative sign between amplitudes with anti-fermions and without. We verify the correctness of the transformation laws by a direct computation of amplitudes with a massive pair of quarks or scalars by using Berends-Giele type recursion relations and the diagrammatic rules of [29].

This article is organized as follows. In section 2 we review the notation for colour ordered partial amplitudes and the conventions for spinors and polarization vectors. In section 3 we introduce our supersymmetric model, give the supersymmetric transformation laws and apply them to derive relations among various amplitudes. In section 4 we consider as an example amplitudes with a massive pair of quarks or scalars plus an arbitrary number of positive helicity gluons. Our conclusions are contained in section 5.

In appendix A we list our sign conventions. Appendix B contains the derivation of the supersymmetric transformation laws. In appendix C we extend the diagrammatic rules obtained in [29] to the supersymmetric model. In appendix D we give some technical details on the solution of the recurrence relation discussed in section 4.

2 Review of notation and off-shell continuation

2.1 Colour decomposition

In this work we are concerned with the evaluation of helicity amplitudes with one massive quark pair. We employ the usual decomposition of the full amplitude \mathcal{A}_n into gauge invariant partial amplitudes A_n defined by [7, 32, 33]:

$$\mathcal{A}_n(\bar{Q}_1, g_2, g_3, \dots, g_{n-1}, Q_n) = \sum_{\sigma \in S_{n-2}(2, \dots, n-1)} (T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n-1)}})_{ij} A_n(\bar{Q}_1, g_{\sigma(2)}, \dots, g_{\sigma(n-1)}, Q_n), \quad (2.1)$$

where the sum is over all permutations of the external gluon legs. The colour structure is contained in group theoretical factors given by traces over the generators T^a of the fundamental representation of $SU(3)$.

We will also encounter amplitudes with scalars transforming under the fundamental representation of $SU(3)$. The colour decomposition is similar to the one in eq. (2.1):

$$\mathcal{A}_n(\phi_1, g_2, g_3, \dots, g_{n-1}, \phi_n) =$$

$$g^{n-2} \sum_{\sigma \in S_{n-2}(2, \dots, n-1)} (T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n-1)}})_{ij} A_n (\phi_1, g_{\sigma(2)}, \dots, g_{\sigma(n-1)}, \phi_n). \quad (2.2)$$

2.2 Polarization vectors and spinors

It is a well known fact that any four-vector can be decomposed into a sum of two light-like four-vectors. Given a fixed light-like four-vector q , one can associate to any four-vector k (which need not be light-like) a light-like four-vector k^\flat , defined by

$$k^\mu = k^\flat \mu + \frac{k^2}{2(k \cdot q)} q^\mu. \quad (2.3)$$

This prescription has also been used for the off-shell continuation of MHV amplitudes [34] in the context of the CSW rules. Unless stated otherwise, we will use the same reference vector q for all momenta. The projected four-vector k^\flat can be used to define continuations off the null-space of spinor products as follows:

$$\langle k_1 k_2 \rangle = \langle k_1^\flat - |k_2^\flat + \rangle, \quad [k_1 k_2] = \langle k_1^\flat + |k_2^\flat - \rangle. \quad (2.4)$$

The two two-component spinors $|k^\flat + \rangle$ and $|k^\flat - \rangle$ are the Weyl spinors corresponding to the projected momentum $k^\flat \mu$. These can be used to define the off-shell continuation of the polarization vectors for gluons:

$$\epsilon_\mu^\pm(k, q) = \pm \frac{\langle q \mp | \gamma_\mu | k^\flat \mp \rangle}{\sqrt{2} \langle q \mp | k^\flat \pm \rangle}. \quad (2.5)$$

Helicity methods can be extended to massive fermions [17, 31, 35, 36]. We introduce an off-shell continuation of massive spinors, using the projection (2.3) as for the gluon polarization vectors:

$$\begin{aligned} u(k, \pm) &= \frac{\not{k} + m}{\langle k^\flat \mp | q^\pm \rangle} |q^\pm \rangle = |k^\flat \mp \rangle + \frac{m}{\langle k^\flat \mp | q^\pm \rangle} |q^\pm \rangle, \\ \bar{u}(k, \pm) &= \langle q \mp | \frac{\not{k} + m}{\langle q \mp | k^\flat \pm \rangle} = \langle k^\flat \pm | + \frac{m}{\langle q \mp | k^\flat \pm \rangle} \langle q \mp |, \\ v(k, \pm) &= \frac{\not{k} - m}{\langle k^\flat \mp | q^\pm \rangle} |q^\pm \rangle = |k^\flat \mp \rangle - \frac{m}{\langle k^\flat \mp | q^\pm \rangle} |q^\pm \rangle, \\ \bar{v}(k, \pm) &= \langle q \mp | \frac{\not{k} - m}{\langle q \mp | k^\flat \pm \rangle} = \langle k^\flat \pm | - \frac{m}{\langle q \mp | k^\flat \pm \rangle} \langle q \mp |. \end{aligned} \quad (2.6)$$

The normalization is chosen in order to allow for a smooth massless limit. We label the helicities as if all particles were outgoing. As a consequence, the spinors $u(k)$ and $\bar{v}(k)$, which correspond to particles with incoming momentum, have a reversed helicity assignment.

The reference momentum used in the definition (2.6) is not an unphysical quantity that has to drop out in the final result for the helicity amplitudes, as in the case of the

definition of the gluon polarization vectors, but rather defines the quantization axis of the quark spin [35]. Since for any given choice of q the spinors (2.6) form a complete basis of solutions of the Dirac equation, the calculation can be simplified without loss of generality by a suitable choice of q . The amplitudes for a desired physical polarization can be obtained by a straightforward linear transformation.

3 SUSY Ward identities for massive quarks

Calculating examples of helicity amplitudes for a massive quark pair and positive helicity gluons, one observes a simple relation to amplitudes with massive scalars that have been calculated in [25, 27, 37]. As a simple example, we find the four point function

$$A(\bar{Q}_1^+, g_2^+, g_3^+, Q_4^-) = \frac{\langle 4q \rangle [23]}{\langle 1q \rangle \langle 23 \rangle} \frac{2im^2}{(k_1 + k_2)^2 - m^2}. \quad (3.1)$$

The results for an arbitrary number of positive helicity gluons are quoted in appendix D. Comparing to the scattering amplitude for two massive scalars and two positive helicity gluons [25, 27, 37]

$$A(\bar{\phi}_1^+, g_2^+, g_3^+, \phi_4^-) = \frac{[23]}{\langle 23 \rangle} \frac{2im^2}{(k_1 + k_2)^2 - m^2}, \quad (3.2)$$

we see these amplitudes are related by a factor $\frac{\langle 1q \rangle}{\langle 4q \rangle}$, similar to the SUSY relation between quark and gluon amplitudes. In the next subsection we define a supersymmetric toy model, which contains QCD as a subset, and derive the supersymmetric transformation laws. In subsection 3.3 we apply these relations to amplitudes involving a pair of massive quarks or scalars. It turns out, that the relations are particular simple, if the reference spinor defining the spin axis of the heavy quarks, is chosen to be same for both the quark and the anti-quark. In subsection 3.4 we discuss modifications which arise if the reference spinors are not identical.

3.1 Definition of the model

To embed QCD with massive quarks into a supersymmetric extension, we have to use two chiral super-multiplets $\Phi_+ = (\varphi_+, \psi_+, F_+)$ and $\bar{\Phi}_- = (\bar{\varphi}_-, \bar{\psi}_-, \bar{F}_-)$ and combine the spinor component fields into Dirac spinors as follows

$$\bar{\Psi} = (\psi_-, \bar{\psi}_+), \quad \Psi = \begin{pmatrix} \psi_+ \\ \bar{\psi}_- \end{pmatrix}. \quad (3.3)$$

The chiral fields are written in component fields as follows:

$$\begin{aligned} \Phi_+(y, \theta) &= \varphi_+(y) + \sqrt{2}\theta\psi_+(y) + \theta^2F_+(y), & y^\mu &= x^\mu - i\theta\sigma^\mu\bar{\theta}, \\ \bar{\Phi}_-(\bar{y}, \bar{\theta}) &= \bar{\varphi}_-(\bar{y}) + \sqrt{2}\bar{\theta}\bar{\psi}_-(\bar{y}) + \bar{\theta}^2\bar{F}_-(\bar{y}), & \bar{y}^\mu &= x^\mu - i\bar{\theta}\bar{\sigma}^\mu\theta. \end{aligned} \quad (3.4)$$

The Lagrange density of our model is given by

$$\begin{aligned}\mathcal{L} = & \frac{1}{8g^2} \text{Tr } WW|_F + \frac{1}{8g^2} \text{Tr } \bar{W}\bar{W}|_{\bar{F}} \\ & + \bar{\Phi}_+ e^{-2gV} \Phi_+|_D + \bar{\Phi}_- e^{2gV^T} \Phi_-|_D + m\Phi_- \Phi_+|_F + m\bar{\Phi}_+ \bar{\Phi}_-|_{\bar{F}},\end{aligned}\quad (3.5)$$

where the vector multiplet $V^a = (A^a, \lambda^a, D^a)$ contains the gluon A^a and the gluino λ^a . The field strength is defined by

$$W_A = -\frac{1}{4} (\bar{D}\bar{D}) (e^{2gV} D_A e^{-2gV}), \quad \bar{W}_{\dot{A}} = -\frac{1}{4} (DD) ((\bar{D}_{\dot{A}} e^{-2gV}) e^{2gV}), \quad V = T^a V^a, \quad (3.6)$$

and V^a is given in the Wess-Zumino gauge by

$$V^a = (\theta \sigma^\mu \bar{\theta}) A_\mu^a + i\theta^2 (\bar{\theta} \bar{\lambda}^a) - i\bar{\theta}^2 (\theta \lambda^a) + \frac{1}{2} \theta^2 \bar{\theta}^2 D^a. \quad (3.7)$$

For the gluino field we define a four-component spinor as follows:

$$\bar{\Lambda} = (i\lambda^A, -i\bar{\lambda}_{\dot{A}}), \quad \Lambda = \begin{pmatrix} i\lambda_A \\ -i\bar{\lambda}^{\dot{A}} \end{pmatrix}. \quad (3.8)$$

It is also convenient to redefine the scalar fields as follows:

$$\bar{\phi}_- = \varphi_-, \quad \phi_+ = \bar{\varphi}_-, \quad \bar{\phi}_+ = \bar{\varphi}_+, \quad \phi_- = \varphi_+. \quad (3.9)$$

Eliminating the D - and F -terms and with the convention that

$$(\bar{\phi}_\pm)^\dagger = \phi_\mp \quad (3.10)$$

we obtain

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4} (F_{\mu\nu}^a)^2 + \frac{i}{2} \bar{\Lambda}^a \gamma^\mu D_\mu^{ab} \Lambda^b + i\bar{\Psi} \gamma^\mu D_\mu \Psi - m\bar{\Psi} \Psi \\ & + (D_\mu \phi_+)^{\dagger} (D^\mu \phi_+) - m^2 \bar{\phi}_- \phi_+ + (D_\mu \phi_-)^{\dagger} (D^\mu \phi_-) - m^2 \bar{\phi}_+ \phi_- \\ & - \sqrt{2}g [\bar{\phi}_+ \bar{\Lambda}^a T^a P_+ \Psi + \bar{\Psi} P_- \Lambda^a T^a \phi_- - \bar{\phi}_- \bar{\Lambda}^a T^a P_- \Psi - \bar{\Psi} P_+ \Lambda^a T^a \phi_+] \\ & - \frac{1}{2} g^2 (\bar{\phi}_+ T^a \phi_- - \bar{\phi}_- T^a \phi_+)^2,\end{aligned}\quad (3.11)$$

where the covariant derivative in the fundamental and adjoint representation is given by

$$D_\mu = \partial_\mu - igT^a A_\mu^a, \quad D_\mu^{ab} = \partial_\mu - g f^{abc} A_\mu^c. \quad (3.12)$$

The chiral projectors are denoted as usual by $P_\pm = 1/2(1 \pm \gamma_5)$. As in [29] we can derive primitive vertices for the additional interactions that arise compared to nonsupersymmetric QCD. The results are collected in appendix C.

3.2 Transformation of the fields

The SUSY transformations of helicity states have been first derived in [13] and later applied to QCD helicity amplitudes [14, 16]. We use a commuting SUSY generator $Q_{\text{SUSY}}(\eta) = \eta^A Q_A + \bar{\eta}_A \bar{Q}^A$ obtained by multiplying the SUSY generators by a Grassmann valued spinor η . This generator satisfies the SUSY algebra

$$[Q_{\text{SUSY}}(\eta), Q_{\text{SUSY}}(\xi)] = -2i\bar{\eta}\bar{\partial}\xi. \quad (3.13)$$

SUSY transformations of generic bosonic or fermionic fields Ψ are generated by $\delta_\eta \Psi = [Q_{\text{SUSY}}(\eta), \Psi]$. With the notation

$$\Gamma_\eta^\pm(k) = \sqrt{2} \langle \eta^\pm | k^\mp \rangle, \quad \Sigma_\eta^\pm(k, q) = \sqrt{2}m \frac{\langle q^\pm | \eta^\mp \rangle}{\langle q^\pm | k^\mp \rangle} \quad (3.14)$$

one finds the transformation laws for outgoing particles:

$$\begin{aligned} \delta_\eta \bar{Q}^\pm &= \Gamma_\eta^\pm(k) \bar{\phi}^\pm - \Sigma_\eta^\mp(k, q) \bar{\phi}^\mp, & \delta_\eta \bar{\phi}^\pm &= \Gamma_\eta^\mp(k) \bar{Q}^\pm + \Sigma_\eta^\mp(k, q) \bar{Q}^\mp, \\ \delta_\eta Q^\pm &= -\Gamma_\eta^\pm(k) \phi^\pm + \Sigma_\eta^\mp(k, q) \phi^\mp, & \delta_\eta \phi^\pm &= -\Gamma_\eta^\mp(k) Q^\pm - \Sigma_\eta^\mp(k, q) Q^\mp. \end{aligned} \quad (3.15)$$

Details on the derivation are given in appendix B. Eq. (3.15) is the main result of this paper and generalizes the well-known results for massless particles to the massive case. Compared to the massless case there are additional terms with Σ^\pm , which are proportional to the mass of the particles in the super-multiplet. For practical applications, one can achieve a considerable simplification by noting that $\Sigma_q(k, q) = 0$. Therefore the additional contributions to (3.15) can be avoided *provided* the SUSY spinors $|\eta^\pm\rangle$ are taken to be equal to a Grassmann parameter θ times the reference spinors $|q^\pm\rangle$ used in the definition of the massive spinors (2.6). This requires that the same reference spinors $|q^\pm\rangle$ are chosen for all massive quarks in the amplitude. In this case we have the following transformation laws, which are identical with the massless case:

$$\begin{aligned} \delta_\eta \bar{Q}^\pm &= \Gamma_\eta^\pm(k) \bar{\phi}^\pm, & \delta_\eta \bar{\phi}^\pm &= \Gamma_\eta^\mp(k) \bar{Q}^\pm, \\ \delta_\eta Q^\pm &= -\Gamma_\eta^\pm(k) \phi^\pm, & \delta_\eta \phi^\pm &= -\Gamma_\eta^\mp(k) Q^\pm. \end{aligned} \quad (3.16)$$

The gluon and the gluino transform as follows:

$$\begin{aligned} \delta_\eta g^\pm &= -\Gamma_\eta^\pm(k) \bar{\Lambda}^\pm = \Gamma_\eta^\pm(k) \Lambda^\pm, \\ \delta_\eta \bar{\Lambda}^\pm &= -\Gamma_\eta^\mp(k) g^\pm, \quad \delta_\eta \Lambda^\pm = \Gamma_\eta^\mp(k) g^\pm. \end{aligned} \quad (3.17)$$

3.3 SUSY Ward identities for massive quarks and scalars

We can now derive relations connecting amplitudes of massive quarks to amplitudes of massive scalars, generalizing the example of the four point amplitudes (3.1) and (3.2). In this section we will always choose $\eta = \theta q$ and use the same reference vector q in the

definition of the spinors for all massive quarks. Therefore terms proportional to Σ^\pm do not contribute. Furthermore we treat in this section all particles as outgoing.

Since the SUSY charge Q_{SUSY} annihilates the vacuum for unbroken SUSY, we obtain SUSY Ward Identities (SWIs) from the relation [13]

$$0 = \langle 0 | [Q_{\text{SUSY}}(\eta), z_1 \dots z_n] | 0 \rangle = \sum_i \langle 0 | z_1 \dots [Q_{\text{SUSY}}(\eta), z_i] \dots z_n | 0 \rangle, \quad (3.18)$$

where the z_i are arbitrary creation or annihilation operators.

Applying the SUSY transformation to the amplitude

$$A(\bar{\phi}_1^+, g_2^+, \dots, g_{n-1}^+, Q_n^-) \quad (3.19)$$

with one $\bar{\phi}^+$ scalar, $(n-2)$ positive helicity gluons and a negative helicity quark results in the relation

$$\begin{aligned} \Gamma_\eta^-(k_1) A(\bar{Q}_1^+, g_2^+, \dots, g_{n-1}^+, Q_n^-) - \sum_{i=2}^{n-1} \Gamma_\eta^+(k_i) A(\bar{\phi}_1^+, g_2^+, \dots, \bar{\Lambda}_i^+, \dots, g_{n-1}^+, Q_n^-) \\ - \Gamma_\eta^-(k_n) A(\bar{\phi}_1^+, g_2^+, \dots, g_{n-1}^+, \phi_n^-) = 0. \end{aligned} \quad (3.20)$$

Note that these identities hold to all orders in perturbation theory in the supersymmetric toy model. However, only on tree level the scalar and quark amplitudes are identical to those in pure QCD while in higher orders loops involving SUSY particles contribute.

The terms arising from the transformations of the gluons vanish at tree level since there are no primitive vertices of the form $V(\bar{\phi}_1^+, \bar{\Lambda}_i^+, Q_n^-)$. Therefore we obtain the relation

$$A_n(\bar{Q}_1^+, g_2^+, \dots, g_{n-1}^+, Q_n^-) = \frac{\langle nq \rangle}{\langle 1q \rangle} A_n(\bar{\phi}_1^+, g_2^+, \dots, g_{n-1}^+, \phi_n^-), \quad (3.21)$$

which generalizes the relation among the four point amplitudes (3.1) and (3.2) to an arbitrary number of positive helicity gluons. In the same way, the SUSY transformation of $A(\bar{Q}_1^-, g_2^+, \dots, g_{n-1}^+, \phi_n^+)$ gives the corresponding relation for exchanged quark helicities:

$$A_n(\bar{Q}_1^-, g_2^+, \dots, g_{n-1}^+, Q_n^+) = -\frac{\langle 1q \rangle}{\langle nq \rangle} A_n(\bar{\phi}_1^-, g_2^+, \dots, g_{n-1}^+, \phi_n^+). \quad (3.22)$$

We can also derive SWIs for amplitudes including a negative helicity gluon. Leaving the helicity of one massive quark arbitrary, these relations read

$$\begin{aligned} 0 = \delta_\eta A(\bar{\phi}_1^+, g_2^+, \dots, g_j^-, \dots, Q_n^\pm) = \Gamma_\eta^-(k_1) A(\bar{Q}_1^+, g_2^+, \dots, g_j^-, \dots, Q_n^\pm) \\ - \Gamma_\eta^-(k_j) A(\bar{\phi}_1^+, g_2^+, \dots, \bar{\Lambda}_j^-, \dots, Q_n^\pm) - \Gamma_\eta^\pm(k_n) A(\bar{\phi}_1^+, g_2^+, \dots, g_j^-, \dots, \phi_n^\pm), \end{aligned} \quad (3.23a)$$

where the terms from the transformation of the positive helicity gluons vanish for the same reason as above. For the amplitude with two positive helicity quarks $A(\bar{Q}_1^+, \dots, g_j^-, \dots, Q_n^+)$,

the last term vanishes since it involves two different scalars $\bar{\phi}^+$ and ϕ^+ . This ‘helicity flip’ amplitude is therefore directly related to an amplitude involving a gluino and a scalar.

$$A(\bar{Q}_1^+, g_2^+, \dots, g_j^-, \dots, Q_n^+) = \frac{\langle j q \rangle}{\langle 1 q \rangle} A(\bar{\phi}_1^+, g_2^+, \dots, \bar{\Lambda}_j^-, \dots, Q_n^+). \quad (3.23b)$$

For the amplitude with one negative helicity quark, all three contributions survive. The corresponding identity for the opposite quark helicities is given by

$$0 = \delta_\eta A(\bar{Q}_1^\pm, g_2^+, \dots, g_j^-, \dots, \phi_n^+) = \Gamma_\eta^\pm(k_1) A(\bar{\phi}_1^\pm, g_2^+, \dots, g_j^-, \dots, \phi_n^+) \\ - \Gamma_\eta^-(k_j) A(\bar{Q}_1^\pm, g_2^+, \dots, \Lambda_j^-, \dots, \phi_n^+) + \Gamma_\eta^-(k_n) A(\bar{Q}_1^\pm, g_2^+, \dots, g_j^-, \dots, Q_n^+). \quad (3.23c)$$

We can simplify the identities (3.23) further by a specific choice of the reference spinors. With the choice $|q+\rangle = |j+\rangle$ we obtain

$$A(\bar{Q}_1^+, g_2^+, \dots, g_j^-, \dots, Q_n^+) \Big|_{|q+\rangle=|j+\rangle} = 0, \quad (3.24)$$

$$A(\bar{Q}_1^+, g_2^+, \dots, g_j^-, \dots, Q_n^-) \Big|_{|q+\rangle=|j+\rangle} = \frac{\langle nj \rangle}{\langle 1j \rangle} A_n(\bar{\phi}_1^+, g_2^+, \dots, g_j^-, \dots, \phi_n^-), \quad (3.25)$$

$$A(\bar{Q}_1^-, g_2^+, \dots, g_j^-, \dots, Q_n^+) \Big|_{|q+\rangle=|j+\rangle} = -\frac{\langle 1j \rangle}{\langle nj \rangle} A_n(\bar{\phi}_1^-, g_2^+, \dots, g_j^-, \dots, \phi_n^+). \quad (3.26)$$

The scalar amplitudes appearing in these identities have been computed for an arbitrary number of gluons in [27] using on-shell recursion relations. However it should be noted that the choice $|q+\rangle = |j+\rangle$ defines the quantization axis of the spins of the heavy quarks.

Finally we can consider amplitudes with two negative helicity quarks and positive helicity gluons only. In this case the SWI relates the quark amplitude to a sum of gluino amplitudes:

$$0 = \delta_\eta A(\bar{\phi}_1^-, g_2^+, \dots, g_{n-1}^+, Q_n^-) = \Gamma_\eta^+(k_1) A(\bar{Q}_1^-, g_2^+, \dots, g_{n-1}^+, Q_n^-) \\ - \sum_{i=2}^{n-1} \Gamma_\eta^+(k_i) A(\bar{\phi}_1^-, g_2^+, \dots, \bar{\Lambda}_i^+, \dots, g_{n-1}^+, Q_n^-), \quad (3.27)$$

where the additional term from the transformation of the quark vanishes.

These last examples show that SUSY methods can be extended to amplitudes with additional negative helicity partons. However in general these relations will involve amplitudes with external gluinos.

3.4 Alternative choice of reference spinors

In [31] a different choice of auxiliary spinors than in (2.6) has been used to calculate helicity amplitudes for a massive quark pair and an arbitrary number of positive helicity gluons using the Berends-Giele relations. In this subsection we show how to relate our results to

those of [31]. In that work the momenta of the massive quarks k_1 and k_n are decomposed in terms of two lightlike vectors $\hat{k}_{1/n}$ according to

$$k_1^\mu = \beta_+ \hat{k}_1^\mu + \beta_- \hat{k}_n^\mu, \quad k_n^\mu = \beta_- \hat{k}_1^\mu + \beta_+ \hat{k}_n^\mu, \quad (3.28)$$

with $\beta_\pm = \frac{1}{2}(1 \pm \beta)$ where $\beta = \sqrt{1 - \frac{4m^2}{(k_1+k_n)^2}}$ is the quark velocity. The spinors are then defined similarly to (2.6) but using $\beta_+ \hat{k}_n$ as reference momentum for \bar{Q}_1 and $\beta_+ \hat{k}_1$ as reference momentum for Q_n . Explicitly they read:

$$\begin{aligned} \bar{u}(1, \pm) &= \langle \hat{n} \mp | \frac{\not{k}_1 + m}{\beta_+^{1/2} \langle \hat{n} \mp | \hat{1} \pm \rangle} = \beta_+^{1/2} \langle \hat{1} \pm | + \frac{m}{\beta_+^{1/2} \langle \hat{n} \mp | \hat{1} \pm \rangle} \langle \hat{n} \mp |, \\ v(n, \pm) &= \frac{\not{k}_n - m}{\beta_+^{1/2} \langle \hat{n} \mp | \hat{1} \pm \rangle} | \hat{1} \pm \rangle = \beta_+^{1/2} | \hat{n} \mp \rangle - \frac{m}{\beta_+^{1/2} \langle \hat{n} \mp | \hat{1} \pm \rangle} | \hat{1} \pm \rangle. \end{aligned} \quad (3.29)$$

Since the choice of different reference spinors for the two quarks corresponds to different spin-vectors, different helicity combinations give a nonvanishing result compared to our formalism where the same reference spinor is used for both quarks. The general transformations (3.15) still relate amplitudes with massive quarks to amplitudes with massive scalars. As an example we consider the SWI obtained from

$$\delta_\eta A(\bar{\phi}_1^+, g_2^+, \dots, g_{n-1}^+, Q_n^+) = 0. \quad (3.30)$$

The explicit forms of the relevant transformation formulae from (3.15) read:

$$\begin{aligned} \delta_\eta Q_n^+ &= -\sqrt{2} \beta_+^{1/2} \langle \eta + | \hat{n} - \rangle \phi_n^+ + \sqrt{2} m \beta_+^{-1/2} \frac{\langle \hat{1} - | \eta + \rangle}{\langle \hat{1} - | \hat{n} + \rangle} \phi_n^-, \\ \delta_\eta \bar{\phi}_1^+ &= \sqrt{2} \beta_+^{1/2} \langle \eta - | \hat{1} + \rangle \bar{Q}_1^+ + \sqrt{2} m \beta_+^{-1/2} \frac{\langle \hat{n} - | \eta + \rangle}{\langle \hat{n} - | \hat{1} + \rangle} \bar{Q}_1^-, \end{aligned} \quad (3.31)$$

and by choosing $\eta = \theta \hat{k}_n$ we obtain for the helicity flip amplitude

$$A(\bar{Q}_1^+, g_2^+, \dots, g_{n-1}^+, Q_n^+) = \frac{m}{\beta_+ \langle \hat{1} - | \hat{n} + \rangle} A(\bar{\phi}_1^+, g_2^+, \dots, g_{n-1}^+, \phi_n^-). \quad (3.32)$$

This relation can immediately be verified for $n = 4$: Ref [31] obtains for the four point function the result

$$A(\bar{Q}_1^+, g_2^+, g_3^+, Q_4^+) = \frac{m}{\beta_+ \langle \hat{1} - | \hat{4} + \rangle} \frac{[23]}{\langle 23 \rangle} \frac{2im^2}{(k_1 + k_2)^2 - m^2}. \quad (3.33)$$

The scalar amplitude is given by [27]

$$A(\bar{\phi}_1^+, g_2^+, g_3^+, \phi_4^-) = \frac{[23]}{\langle 23 \rangle} \frac{2im^2}{(k_1 + k_2)^2 - m^2}. \quad (3.34)$$

4 Recursion relation and results for amplitudes with positive helicity gluons

In this section we consider the degree zero amplitudes

$$A(\bar{Q}_1^+, g_2^+, \dots, g_{n-1}^+, Q_n^-) \quad \text{and} \quad A(\bar{\phi}_1^+, g_2^+, \dots, g_{n-1}^+, \phi_n^-) \quad (4.1)$$

with a massive pair of quarks or scalars plus an arbitrary number of positive helicity gluons. In the following we will discuss amplitudes with one particle off-shell, this particle will be marked by a hat. It is also convenient to treat Q_n as an incoming quark rather than an outgoing anti-quark. Let us take the outgoing particles 1 to $n-1$ with positive energy. Therefore $p_n = k_1 + \dots + k_{n-1} = -k_n$ has a positive energy component. With our phase conventions, the corresponding amplitudes are related by

$$A(\bar{Q}_1^+, g_2^+, \dots, g_{n-1}^+, Q_{k_n}^-) = iA(\bar{Q}_1^+, g_2^+, \dots, g_{n-1}^+ | Q_{p_n=-k_n}^-). \quad (4.2)$$

In the following we will use the diagrammatic rules of [29] that are briefly reviewed in appendix C. The amplitudes in eq.(4.1) are of degree zero, therefore in our conventions no helicity-flip vertex occurs. As a consequence, the corresponding amplitudes with one-leg off-shell satisfy rather simple Berends-Giele type recurrence relations:

$$A_n(\bar{Q}_1^+, \dots, g_{n-1}^+ | \hat{Q}_{p_n}^-) = \sum_{j=2}^{n-1} V_3(\bar{Q}_{1,j}^+, g_{j,n}^+, Q_{p_n}^-) \frac{i}{k_{1,j}^2 - m^2} A_j(\bar{Q}_1^+, \dots, g_{j-1}^+ | \hat{Q}_{1,j}^-) \\ \frac{i}{k_{j,n}^2} A_{n-j+1}(g_j^+, \dots, g_{n-1}^+, \hat{g}_{-(j,n)}^-), \quad (4.3)$$

where we have defined

$$k_{i,j} = \sum_{l=i}^{j-1} k_l. \quad (4.4)$$

The corresponding recurrence relation for the scalars reads:

$$A_n(\bar{\phi}_1^+, \dots, g_{n-1}^+, \hat{\phi}_n^-) = \sum_{j=2}^{n-1} V_3(\bar{\phi}_{1,j}^+, g_{j,n}^+, \phi_n^-) \frac{i}{k_{1,j}^2 - m^2} A_j(\bar{\phi}_1^+, \dots, g_{j-1}^+, \hat{\phi}_{-(1,j)}^-) \\ \frac{i}{k_{j,n}^2} A_{n-j+1}(g_j^+, \dots, g_{n-1}^+, \hat{g}_{-(j,n)}^-). \quad (4.5)$$

Here the quartic primitive scalar-gluon vertices do not contribute since the off-shell continuation of the polarization vectors (2.5) implies $\epsilon^+(k_i, q) \cdot \epsilon^+(k_j, q) = 0$.

The recursion starts with the two-point amplitudes, which are given by the inverse propagators:

$$A_2(g_j^+, \hat{g}_{-j}^-) = -ik_j^2, \quad A_2(\bar{Q}_1^+ | \hat{Q}_1) = A_2(\bar{\phi}_1^+, \hat{\phi}_{-1}) = -i(k_1^2 - m^2). \quad (4.6)$$

These recurrence relations are solved explicitly in appendix D. In the present example we can use the recursion relation to establish the relation (3.21) between the amplitudes of quarks and scalars directly.

After applying crossing symmetry the quark amplitude entering the SWI can be written as

$$\begin{aligned}
A_n(\bar{Q}_1^+, \dots, g_{n-1}^+, \hat{Q}_n^-) &= i A_n(\bar{Q}_1^+, \dots, g_{n-1}^+ | \hat{Q}_{p_n=-k_n}^-) \\
&= i \sum_{j=2}^{n-1} V_3(\bar{Q}_{1,j}^+, g_{j,n}^+, Q_{p_n}^-) \frac{i}{k_{1,j}^2 - m^2} \\
&\quad (-i) A_j(\bar{Q}_1^+, \dots, g_{j-1}^+, \hat{Q}_{-(1,j)}^-) \frac{i}{k_{j,n}^2} A_{n-j+1}(g_j^+, \dots, g_{n-1}^+, \hat{g}_{-(j,n)}^-). \quad (4.7)
\end{aligned}$$

We now use the induction assumption that the identity (3.21) holds up to $n-1$ already for the one-leg off-shell amplitudes. This is true for $n=2$:

$$A_2(\bar{Q}_1^+, \hat{Q}_2^-) = i A_2(\bar{\phi}_1^+, \hat{\phi}_2^-). \quad (4.8)$$

Since the primitive cubic scalar-gluon vertex (C.3) and the quark gluon vertex (C.1) satisfy the identity

$$V(\bar{Q}_k^+, l^+, Q_{p_n=-k_n}^-) = \frac{\langle p_n q \rangle}{\langle k q \rangle} V(\bar{\phi}_k^+, l^+, \phi_n^-) = -i \frac{\langle n q \rangle}{\langle k q \rangle} V(\bar{\phi}_k^+, l^+, \phi_n^-), \quad (4.9)$$

we find that every term of the sum in the recursion relation contains a product of the form

$$\begin{aligned}
&V_3(\bar{Q}_{1,j}^+, g_{j,n}^+, Q_{p_n}^-) A_j(\bar{Q}_1^+, \dots, g_{j-1}^+, \hat{Q}_{-(1,j)}^-) \\
&= \left(-i \frac{\langle n q \rangle}{\langle (1,j) q \rangle} V_3(\bar{\phi}_{1,j}^+, g_{j,n}^+, \phi_n^-) \right) \left(\frac{\langle -(1,j) q \rangle}{\langle 1 q \rangle} A_j(\bar{\phi}_1^+, \dots, (j-1)^+, \hat{\phi}_{-(1,j)}^-) \right) \\
&= \frac{\langle n q \rangle}{\langle 1 q \rangle} V_3(\bar{\phi}_{1,j}^+, g_{j,n}^+, \phi_n^-) A_j(\bar{\phi}_1^+, \dots, (j-1)^+, \hat{\phi}_{-(1,j)}^-) \quad (4.10)
\end{aligned}$$

Therefore in every term appears the same overall-factor $\frac{\langle n q \rangle}{\langle 1 q \rangle}$ that can be pulled out of the sum. The remaining factors give the scalar amplitude (4.5) and we obtain the SWI (3.21):

$$A(\bar{Q}_1^+, g_2^+, \dots, g_{n-1}^+, Q_n^-) = \frac{\langle n q \rangle}{\langle 1 q \rangle} A(\bar{\phi}_1^+, g_2^+, \dots, g_{n-1}^+, \phi_n^-). \quad (4.11)$$

This is a non-trivial cross-check on the correctness of the supersymmetric transformation laws in eq.(3.15), in particular since the equivalence of the analytic expressions for the all-multiplicity quark amplitudes [31] with the scalar amplitudes [27] is not immediate.

5 Conclusions

In this paper we considered relations based on supersymmetry between multi-gluon amplitudes with a pair massive quarks and corresponding amplitudes where the quarks are replaced by massive scalars. These relations generalize the well-known formulae for the massless case. The relations are useful as they allow to obtain amplitudes with heavy quarks from amplitudes with scalars. Furthermore, if both sets of amplitudes are known, the supersymmetric relations provide a valuable cross-check.

Note added in proof

After the submission of this paper, a form of the scalar and quark amplitudes with an arbitrary number of positive helicity gluons has been found [40] that is much simpler compared to those given in [27, 31] and in (D.7) and that has been used to verify the identity (3.32) explicitly for an arbitrary number of gluons.

Acknowledgments

CS was supported by the DFG through the Graduiertenkolleg "Eichtheorien" at Mainz University and by the DFG Sonderforschungsbereich/Transregio 9 "Computergestützte Theoretische Teilchenphysik".

A Notation and conventions

In this appendix we give a comprehensive summary of our notation and sign conventions. The convention for the metric tensor is $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. We use the Weyl representation for the Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A.1})$$

where $\sigma_{AB}^\mu = (1, -\vec{\sigma})$, $\bar{\sigma}^{\mu AB} = (1, \vec{\sigma})$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. The chiral projectors are as usual $P_\pm = 1/2(1 \pm \gamma_5)$. The two-dimensional antisymmetric tensor is defined by

$$\varepsilon^{AB} = \varepsilon^{\dot{A}\dot{B}} = \varepsilon_{AB} = \varepsilon_{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (\text{A.2})$$

Indices of two-component Weyl spinors are raised and lowered as follows:

$$\psi^A = \varepsilon^{AB}\psi_B, \quad \bar{\psi}^{\dot{A}} = \varepsilon^{\dot{A}\dot{B}}\bar{\psi}_{\dot{B}}, \quad \bar{\psi}_{\dot{B}} = \bar{\psi}^{\dot{A}}\varepsilon_{\dot{A}\dot{B}}, \quad \psi_B = \psi^A\varepsilon_{AB}. \quad (\text{A.3})$$

The generators of supersymmetry are taken to be

$$Q_A = \frac{\partial}{\partial\theta^A} + i\sigma_{A\dot{A}}^\mu \bar{\theta}^{\dot{A}} \partial_\mu, \quad \bar{Q}^{\dot{A}} = \frac{\partial}{\partial\bar{\theta}_{\dot{A}}} + i\bar{\sigma}^{\mu\dot{A}B} \theta_B \partial_\mu. \quad (\text{A.4})$$

They satisfy the supersymmetry algebra:

$$\{Q_A, \bar{Q}_{\dot{A}}\} = -2i\sigma_{A\dot{A}}^\mu \partial_\mu. \quad (\text{A.5})$$

Spinor products are defined as

$$\xi\eta = \xi^A \eta_A, \quad \bar{\xi}\bar{\eta} = \bar{\xi}_{\dot{A}} \bar{\eta}^{\dot{A}}. \quad (\text{A.6})$$

In the bra-ket notation spinor products are denoted as

$$\langle pq \rangle = \langle p- | q+ \rangle = p^A q_A, \quad [qp] = \langle q+ | p- \rangle = q_{\dot{A}} p^{\dot{A}}, \quad (\text{A.7})$$

In terms of the light-cone components

$$p_+ = p_0 + p_3, \quad p_- = p_0 - p_3, \quad p_\perp = p_1 + ip_2, \quad p_{\perp*} = p_1 - ip_2. \quad (\text{A.8})$$

of a null-vector p^μ , the corresponding massless spinors $\langle p \pm |$ and $|p\pm\rangle$ can be chosen as

$$\begin{aligned} |p+\rangle &= \frac{e^{-i\frac{\phi}{2}}}{\sqrt{|p_+|}} \begin{pmatrix} -p_{\perp*} \\ p_+ \end{pmatrix}, & |p-\rangle &= \frac{e^{-i\frac{\phi}{2}}}{\sqrt{|p_+|}} \begin{pmatrix} p_+ \\ p_\perp \end{pmatrix}, \\ \langle p+| &= \frac{e^{-i\frac{\phi}{2}}}{\sqrt{|p_+|}} (-p_\perp, p_+), & \langle p-| &= \frac{e^{-i\frac{\phi}{2}}}{\sqrt{|p_+|}} (p_+, p_{\perp*}), \end{aligned} \quad (\text{A.9})$$

where the phase ϕ is given by

$$p_+ = |p_+| e^{i\phi}. \quad (\text{A.10})$$

If p_+ is real and $p_+ > 0$ we have the following relations between a spinor corresponding to a vector p and a spinor corresponding to a vector $(-p)$:

$$\begin{aligned} |(-p)\pm\rangle &= i|p\pm\rangle, \\ \langle(-p)\pm| &= i\langle p\pm|. \end{aligned} \quad (\text{A.11})$$

Therefore the spinors of massive quarks and anti-quarks are related by $u(-k, \pm) = iv(k, \pm)$ and $\bar{u}(-k, \pm) = i\bar{v}(k, \pm)$. The polarization vectors of the gluons are unchanged under the reversal of the momentum. Denoting a partial amplitude with n incoming particles with momenta p_i and m outgoing particles with momenta k_j by

$$A_{n+m}(\Phi_{k_1} \dots \Phi_{k_m} | \Phi_{p_1} \dots \Phi_{p_n}) \quad (\text{A.12})$$

the amplitude with an outgoing anti-quark is related to that with an incoming quark by

$$A(.., Q_{k_j}, \dots | \dots) = -iA(\dots | \dots, Q_{p_i=-k_j}, \dots) \quad (\text{A.13})$$

where the notation used for the quarks follows that of the spinors, i.e. incoming quarks and outgoing anti-quarks are denoted by Q and outgoing quarks and incoming anti-quarks by \bar{Q} . The same phase as in (A.13) appears for the exchange of an outgoing quark with an incoming anti-quark while for gluons and scalars no phase appears and it is sufficient to reverse the momentum.

B SUSY transformations of helicity states

In this appendix we give the derivation of the supersymmetric transformations in eq. (3.15) and eq. (3.17). We follow closely the approach of ref. [13]. The Lagrangian in eq. (3.5) is invariant under supersymmetric transformations. The supersymmetric transformations act on the component fields of Φ_+ as follows:

$$\begin{aligned}\delta_\eta \varphi_+ &= \sqrt{2}\eta\psi_+, \\ \delta_\eta \psi_+ &= -\sqrt{2}i\sigma^\mu\bar{\eta}\partial_\mu\varphi_+ + \sqrt{2}\eta F_+, \\ \delta_\eta F_+ &= -\sqrt{2}i\bar{\eta}\bar{\sigma}^\mu\partial_\mu\psi_+.\end{aligned}\tag{B.1}$$

Here we used the two-component Weyl spinor notation. The components of $\bar{\Phi}_-$ transform as

$$\begin{aligned}\delta_\eta \bar{\varphi}_- &= \sqrt{2}\bar{\eta}\bar{\psi}_-, \\ \delta_\eta \bar{\psi}_- &= -\sqrt{2}i\bar{\sigma}^\mu\eta\partial_\mu\bar{\varphi}_- + \sqrt{2}\bar{\eta}\bar{F}_-, \\ \delta_\eta \bar{F}_- &= -\sqrt{2}i\eta\sigma^\mu\partial_\mu\bar{\psi}_-.\end{aligned}\tag{B.2}$$

Finally, the components of the vector multiplet transform as

$$\begin{aligned}\delta_\eta A_\mu^a &= i(\bar{\eta}\bar{\sigma}^\mu\lambda^a - \bar{\lambda}^a\bar{\sigma}^\mu\eta), \\ \delta_\eta \lambda^a &= \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)\eta\partial_\mu A_\nu^a + i\eta D^a, \\ \delta_\eta D^a &= \eta\sigma^\mu\partial_\mu\bar{\lambda}^a + \partial_\mu\lambda^a\sigma^\mu\bar{\eta}.\end{aligned}\tag{B.3}$$

We recall the connection between the two-component Weyl spinor notation and the four-component Dirac spinor notation:

$$\begin{aligned}\bar{\Psi} &= (\psi_-, \bar{\psi}_+), & \Psi &= \begin{pmatrix} \psi_+ \\ \bar{\psi}_- \end{pmatrix}, \\ \bar{\Lambda} &= (i\lambda^A, -i\bar{\lambda}_{\dot{A}}), & \Lambda &= \begin{pmatrix} i\lambda_A \\ -i\bar{\lambda}^{\dot{A}} \end{pmatrix}.\end{aligned}\tag{B.4}$$

For the scalars we used the redefinition

$$\bar{\phi}_- = \varphi_-, \quad \phi_+ = \bar{\varphi}_-, \quad \bar{\phi}_+ = \bar{\varphi}_+, \quad \phi_- = \varphi_+.\tag{B.5}$$

together with the convention that

$$(\bar{\phi}_\pm)^\dagger = \phi_\mp.\tag{B.6}$$

The asymptotic fields we may expand in terms of creation and annihilation operators: For the scalars and the quarks we have:

$$\phi_\pm(x) = \int \frac{d^3k}{(2\pi)^3 2k^0} \left(\hat{a}_\mp(k) e^{-ikx} + a_\pm^\dagger(k) e^{ikx} \right),$$

$$\Psi(x) = \sum_{\lambda} \int \frac{d^3 k}{(2\pi)^3 2k^0} (b(k, \lambda) u(k, -\lambda) e^{-ikx} + d^\dagger(k, \lambda) v(k, \lambda) e^{ikx}). \quad (\text{B.7})$$

Let us consider all particles in an amplitude as out-going. To derive supersymmetric relations among different amplitudes with out-going particles we have to know through the reduction formulae the transformation laws of the annihilation operators, e.g

$$\begin{aligned} A(\bar{Q}^\pm(k_1), Q^\pm(k_2), \dots, \bar{\phi}_\pm(k_j), \phi_\pm(k_{j+1}), \dots) = \\ \langle 0 | b(k_1, \pm), d(k_2, \pm), \dots, \hat{a}_\pm(k_j), a_\pm(k_{j+1}), \dots | 0 \rangle. \end{aligned} \quad (\text{B.8})$$

The annihilation operators can be projected out in eq. (B.7): With the notation

$$f \overset{\leftrightarrow}{\partial}_\mu g = f(\partial_\mu g) - (\partial_\mu f) g \quad (\text{B.9})$$

we have

$$\begin{aligned} a_\pm(k) &= i \int d^3 x \left(e^{ikx} \overset{\leftrightarrow}{\partial}_0 \bar{\phi}_\mp(x) \right), & \hat{a}_\pm(k) &= i \int d^3 x \left(e^{ikx} \overset{\leftrightarrow}{\partial}_0 \phi_\mp(x) \right), \\ b(k, \lambda) &= \int d^3 x e^{ikx} \bar{u}(k, \lambda) \gamma^0 \Psi(x), & d(k, \lambda) &= \int d^3 x e^{ikx} \bar{\Psi}(x) \gamma^0 v(k, \lambda). \end{aligned} \quad (\text{B.10})$$

These equations allow us to obtain the transformation laws of the annihilation operators from the known transformation laws of the fields. For example

$$\begin{aligned} \delta_\eta \hat{a}_-(k) &= i \int d^3 x \left(e^{ikx} \overset{\leftrightarrow}{\partial}_0 \delta_\eta \phi_+(x) \right) = i \int d^3 x \left(e^{ikx} \overset{\leftrightarrow}{\partial}_0 \delta_\eta \bar{\varphi}_-(x) \right) \\ &= i \int d^3 x \left(e^{ikx} \overset{\leftrightarrow}{\partial}_0 \sqrt{2} \bar{\eta} \bar{\psi}_-(x) \right) = i \int d^3 x \left(e^{ikx} \overset{\leftrightarrow}{\partial}_0 \sqrt{2} \bar{\eta} P_- \Psi(x) \right). \end{aligned} \quad (\text{B.11})$$

Reinserting the expansion eq. (B.7) for $\Psi(x)$ one finds the transformation laws

$$\begin{aligned} \delta_\eta a_\pm(k) &= \sum_{\lambda} d(k, \lambda) \left(\sqrt{2} \bar{v}(k, -\lambda) P_{\pm} \left(\begin{array}{c} \eta \\ \bar{\eta} \end{array} \right) \right), \\ \delta_\eta \hat{a}_\pm(k) &= \sum_{\lambda} \left(\sqrt{2} (\eta, \bar{\eta}) P_{\pm} u(k, -\lambda) \right) b(k, \lambda). \end{aligned} \quad (\text{B.12})$$

For the fermion field we find

$$\begin{aligned} \delta_\eta b(k, \lambda) &= -\sqrt{2} \bar{u}(k, \lambda) \left(\begin{array}{c} \eta \hat{a}_-(k) \\ \bar{\eta} \hat{a}_+(k) \end{array} \right), \\ \delta_\eta d(k, \lambda) &= \sqrt{2} (a_-(k) \eta, a_+(k) \bar{\eta}) v(k, \lambda). \end{aligned} \quad (\text{B.13})$$

Using the explicit quark spinors in eq. (2.6) and taking the sign change for amplitudes with anti-quarks discussed at the end of this section into account we obtain eq. (3.15). Similar

considerations apply to the particles in the vector multiplet. The gluino is a Majorana fermion with expansion

$$\Lambda(x) = \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3 2k^0} (b(k, \lambda)u(k, -\lambda)e^{-ikx} + b^\dagger(k, \lambda)v(k, \lambda)e^{ikx}). \quad (\text{B.14})$$

The asymptotic expansion of the gluon reads

$$A^\mu(x) = \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3 2k^0} (a(k, \lambda)\varepsilon^\mu(k, \lambda)^*e^{-ikx} + a^\dagger(k, \lambda)\varepsilon^\mu(k, \lambda)e^{ikx}). \quad (\text{B.15})$$

The annihilation operator for the gluon is projected out by

$$a(k, \lambda) = -i \int d^3x e^{ikx} \overleftrightarrow{\partial}_0 \varepsilon_\mu(k, \lambda) A^\mu(x). \quad (\text{B.16})$$

We then obtain

$$\begin{aligned} \delta_\eta a(k, \lambda) &= \varepsilon_\mu(k, \lambda) [\langle \eta - |\gamma^\mu| k- \rangle b(k, -) - \langle \eta + |\gamma^\mu| k+ \rangle b(k, +)], \\ \delta_\eta b(k, \pm) &= \pm \bar{u}(k, \lambda) \gamma^\mu P_{\pm} \left(\frac{\eta}{\bar{\eta}} \right) \sum_{\lambda'} \varepsilon_\mu^*(k, \lambda') a(k, \lambda'). \end{aligned} \quad (\text{B.17})$$

There is one subtlety in writing down the transformation laws for the fields: For the application towards supersymmetric Ward identities we have to keep the relative sign between different amplitudes correct. Let us consider a Feynman graph corresponding to the matrix element

$$\langle 0 | b \dots d \dots T [\bar{\psi}_1 \Gamma_1 \psi_1 \dots \bar{\psi}_n \Gamma_n \psi_n] b^\dagger \dots d^\dagger | 0 \rangle. \quad (\text{B.18})$$

Here $\bar{\psi}_j \Gamma_j \psi_j$ denotes a generic fermion interaction vertex. In order to apply the Feynman rules, we have to anti-commute the fields and the creation and annihilation operators such that

$$\begin{aligned} \langle 0 | \psi b^\dagger(k, \lambda) | 0 \rangle &\rightarrow u(k, \lambda), & \langle 0 | b(k, \lambda) \bar{\psi} | 0 \rangle &\rightarrow \bar{u}(k, \lambda), \\ \langle 0 | \bar{\psi} d^\dagger(k, \lambda) | 0 \rangle &\rightarrow \bar{v}(k, \lambda), & \langle 0 | d(k, \lambda) \psi | 0 \rangle &\rightarrow v(k, \lambda). \end{aligned} \quad (\text{B.19})$$

It is easily seen that this reordering brings a minus sign for each spinor v and \bar{v} [38]. For an individual amplitude these signs are an overall factor and are usually ignored. However, if one relates amplitudes with different particle contents through supersymmetric Ward identities to each other, these signs have to be taken into account. We can incorporate these signs into the transformation laws, by adding an additional sign to each transformation law, for which the number of anti-fermions changes by one unit. For example

$$\delta_\eta g^\pm = -\Gamma_\eta^\pm(k) \bar{\Lambda}^\pm, \quad \delta_\eta g^\pm = \Gamma_\eta^\pm(k) \Lambda^\pm. \quad (\text{B.20})$$

In the second equation we inserted an additional sign with respect to eq.(B.17), which compensates for the change in the number of anti-fermions.

C Diagrammatic rules

In [29] we have obtained diagrammatic rules involving only scalar propagators i/k^2 for gluons and $i/(k^2 - m^2)$ for massive quarks. We introduce so called primitive vertices that are obtained by contracting the vertices of the standard colour ordered Feynman rules with the off-shell polarization vectors and spinors (2.5) and (2.6). Within this approach it is convenient to take the momentum flow of a fermion always in the direction of the fermion arrow line. That implies that we replace an outgoing anti-fermion by an incoming fermion. All other particles remain outgoing. The tri-valent primitive vertices of massive quarks include vertices present both for massless and massive quarks like

$$V_3(\bar{Q}_1^+, Q_2^-, g_3^+) = i\sqrt{2} \frac{[13]^2}{[12]}, \quad V_3(\bar{Q}_1^-, Q_2^+, g_3^-) = i\sqrt{2} \frac{\langle 31 \rangle^2}{\langle 21 \rangle}. \quad (\text{C.1})$$

In addition, there are primitive vertices involving a helicity flip along the quark line that vanish for massless quarks:

$$V_3(\bar{Q}_1^+, Q_2^+, g_3^-) = i\sqrt{2}m \frac{[12]^2}{[23][31]}, \quad V_3(\bar{Q}_1^-, Q_2^-, g_3^+) = -i\sqrt{2}m \frac{\langle 12 \rangle^2}{\langle 23 \rangle \langle 31 \rangle}. \quad (\text{C.2})$$

One can define the degree of a vertex or of an amplitude as the number of “-”-labels minus one. In the diagrammatic rules, only primitive vertices of degree zero and one occur. Furthermore, the degree of an amplitude is exactly the sum of the degrees of the primitive vertices.

It is also straightforward to obtain the primitive vertices involving the scalars and gluinos resulting from the Lagrangian (3.11). For the scalar interactions with gluons we obtain the primitive tri-valent vertices

$$V(\bar{\phi}_1^\pm, \phi_2^\mp, g_3^+) = i\sqrt{2} \frac{[13][23]}{[12]}, \quad V(\bar{\phi}_1^\pm, \phi_2^\mp, g_3^-) = -i\sqrt{2} \frac{\langle 13 \rangle \langle 23 \rangle}{\langle 12 \rangle}. \quad (\text{C.3})$$

The primitive vertices resulting from the couplings between quarks, gluinos and scalars are obtained by inserting the spinors of the massive quarks (2.6). One finds the nonvanishing tri-valent vertices

$$\begin{aligned} V(\bar{Q}_1^+, \phi_2^-, \Lambda_3^+) &= -i\sqrt{2}[13], & V(\bar{Q}_1^-, \phi_2^+, \Lambda_3^-) &= i\sqrt{2} \langle 13 \rangle, \\ V(\bar{Q}_1^-, \phi_2^-, \Lambda_3^+) &= -i\sqrt{2}m \frac{\langle 12 \rangle}{\langle 23 \rangle}, & V(\bar{Q}_1^+, \phi_2^+, \Lambda_3^-) &= i\sqrt{2}m \frac{[12]}{[23]}, \\ V(\bar{\phi}_1^+, Q_2^-, \bar{\Lambda}_3^-) &= -i\sqrt{2} \langle 32 \rangle, & V(\bar{\phi}_1^-, Q_2^+, \bar{\Lambda}_3^+) &= i\sqrt{2}[32], \\ V(\bar{\phi}_1^+, Q_2^+, \bar{\Lambda}_3^-) &= -i\sqrt{2}m \frac{[12]}{[13]}, & V(\bar{\phi}_1^-, Q_2^-, \bar{\Lambda}_3^+) &= i\sqrt{2}m \frac{\langle 12 \rangle}{\langle 13 \rangle}. \end{aligned} \quad (\text{C.4})$$

In addition there are four-valent vertices, which however we do not list here. As for pure QCD only degree one and zero vertices occur. For the application in the SWIs it is important to note that the scalars only couple to gluinos with the opposite helicity label.

D Solution of the recursion relation

In this appendix we give some details on the solution of the recurrence relations (4.3) and (4.5). We follow closely the calculation of [31]. For the solution to the recursion relation (4.3) we make the ansatz

$$A_n(\bar{Q}_1^+, \dots, g_{n-1}^+ | \widehat{Q}_{p_n}^-) = \frac{A_n(\bar{q}_1^+, \dots, g_{n-1}^+ | \widehat{q}_{p_n}^-)}{p_n^2 [(k_1 + k_2)^2 - m^2]} \left[(p_n^2 - m^2) \langle 12 \rangle [21] - m^2 B_n \right], \quad (\text{D.1})$$

where $A_n(\bar{q}_1^+, \dots, g_{n-1}^+ | \widehat{q}_{p_n}^-)$ is the amplitude for *massless* quarks with one leg off-shell [7,32]

$$A_n(\bar{q}_1^+, \dots, g_{n-1}^+ | \widehat{q}_{p_n}^-) = 2^{n/2-1} (-ik_{1,n}^2) \frac{\langle p_n q \rangle}{\langle 12 \rangle \dots \langle (n-2)(n-1) \rangle \langle (n-1)q \rangle}. \quad (\text{D.2})$$

In the massless limit, the formula (D.1) reduces to the known result (D.2) while in the on-shell limit a finite term proportional to m^2 remains that is determined by the quantity B_n :

$$A_n(\bar{Q}_1^+, \dots, g_{n-1}^+ | \widehat{Q}_{p_n}^-) = \frac{im^2 2^{n/2-1}}{(k_1 + k_2)^2 - m^2} \frac{\langle p_n q \rangle B_n}{\langle 12 \rangle \dots \langle (n-1)q \rangle}. \quad (\text{D.3})$$

The structure of the amplitude (D.1) is similar to the current with two off-shell gluons obtained in [39].

In the recursion relation (4.3) also enters the well known expression for the gluon amplitude with one negative helicity gluon [7,8,32,33] and one off-shell leg

$$A_n(g_1^+, \dots, g_{n-1}^+, \widehat{g}_n^-) = 2^{n/2-1} (-ik_{1,n}^2) \frac{\langle qn \rangle^2}{\langle q1 \rangle \langle 12 \rangle \dots \langle (n-2)(n-1) \rangle \langle (n-1)q \rangle}. \quad (\text{D.4})$$

As a check, we have reproduced this amplitude from the Berends-Giele relations within the formalism of [29].

Inserting the ansatz (D.1) into the recursion relation (4.3), one obtains after some tedious steps a recursion relation for B_n that is equivalent to that considered in [31], although there the part vanishing on-shell has been split off differently. Adopting the solution given there, we further split B_n into two contributions

$$B_n = \frac{\langle 12 \rangle}{\langle 1q \rangle (k_{1,4}^2 - m^2)} \left[(k_n^2 - m^2) \langle 2 + |\not{k}_3|q+ \rangle + \tilde{B}_n \right], \quad (\text{D.5})$$

with $\tilde{B}_4 = 0$. The function \tilde{B}_n satisfies the recursion relation

$$\tilde{B}_n = \langle 2 + |\not{k}_1 \not{k}_{2,4} \not{k}_{4,n}|q+ \rangle + \sum_{j=5}^{n-1} \frac{\langle j(j-1) \rangle \langle q - |\not{k}_{1,j} \not{k}_{j,n}|q+ \rangle}{\langle (j-1)q \rangle \langle qj \rangle} \frac{\tilde{B}_j}{(k_{1,j}^2 - m^2)}. \quad (\text{D.6})$$

A closed solution to this equation can be found in [31]. For $n = 4$ we therefore obtain the result quoted in (3.1) while the on-shell amplitudes for $n \geq 5$ are given by

$$A_n(\bar{Q}_1^+, \dots, g_{n-1}^+, Q_n^-) = \frac{im^2 2^{n/2-1}}{(k_{1,3}^2 - m^2)(k_{1,4}^2 - m^2)} \frac{\langle nq \rangle \tilde{B}_n}{\langle 1q \rangle \langle 23 \rangle \dots \langle (n-1)q \rangle}. \quad (\text{D.7})$$

The explicit results for the five and six-point functions obtained from (D.7) and (D.6) are given by

$$A_5(\bar{Q}_1^+, g_2^+, g_3^+, g_4^+, Q_5^-) = 2^{3/3}im^2 \frac{\langle 5q \rangle}{\langle 1q \rangle} \frac{\langle 2 + |\not{k}_1 \not{k}_{2,4}|4- \rangle}{\langle 23 \rangle \langle 34 \rangle (k_{1,3}^2 - m^2)(k_{1,4}^2 - m^2)}, \quad (\text{D.8a})$$

$$A_6(\bar{Q}_1^+, g_2^+, g_3^+, g_4^+, g_5^+, Q_6^-) = 2^{2}im^2 \frac{\langle 6q \rangle}{\langle 1q \rangle \langle 5q \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle (k_{1,3}^2 - m^2)(k_{1,4}^2 - m^2)} \quad (\text{D.8b}) \\ \times \left[\langle 2 + |\not{k}_1 \not{k}_{2,4} \not{k}_{4,6}|q+ \rangle - \frac{\langle q - |\not{k}_{1,5} \not{k}_5|4+ \rangle \langle 2 + |\not{k}_1 \not{k}_{2,4}|4- \rangle}{(k_{1,5}^2 - m^2)} \right].$$

Comparing to the results for scalars obtained in [25, 27, 37] one finds that the five point functions obviously satisfy the SWI (3.21), while the agreement of the six point functions can be established after some use of momentum conservation and Dirac algebra.

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